Statistics 300: Elementary Statistics

Sections 7-2, 7-3, 7-4, 7-5

Parameter Estimation

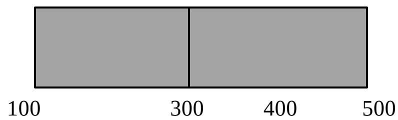
- **Point Estimate**
  - **Best single value to use**
- **Question**
  - **What is the probability this estimate is the correct value?**

Parameter Estimation

- **Question**
  - **What is the probability this estimate is the correct value?**
- **Answer**
  - **zero : assuming "*x*" is a continuous random variable**
  - **Example for Uniform Distribution**

If X ~ U[100,500] then

- P(x = 300) = (300-300)/(500-100)
- = 0



Parameter Estimation

- Pop. mean m
  - Sample mean $\bar{x}$
- Pop. proportion p
  - Sample proportion $\hat{p}$
- Pop. standard deviation s
  - Sample standard deviation s

Problem with Point Estimates

- The unknown parameter (m, p, etc.) is not exactly equal to our sample-based point estimate.
- So, how far away might it be?
- An interval estimate answers this question.

Confidence Interval

- **A range of values that contains the true value of the population parameter with a ...**
- **Specified "level of confidence".**
- **[L(ower limit),U(pper limit)]**

Terminology

- **Confidence Level (a.k.a. Degree of Confidence)**
  - **expressed as a percent (%)**
- **Critical Values (a.k.a. Confidence Coefficients)**

Terminology

- **"alpha" "a" = 1-Confidence**
  - **more about a in Chapter 7**
- **Critical values**
  - **express the confidence level**

Confidence Interval for m
If s is known **(this is a rare situation)**

$$= \overline{x} \pm E$$

$$E = z_{a/2} \cdot \left(\frac{s}{\sqrt{n}} \right)$$

Confidence Interval for m
If s is known **(this is a rare situation)**
if x ~N(?,s)

$$= \overline{x} \pm z_{a/2} \cdot \left(\frac{s}{\sqrt{n}} \right)$$

Why does the
Confidence Interval for m
look like this ?

$$= \overline{x} \pm z_{a/2} \cdot \left(\frac{s}{\sqrt{n}} \right)$$

$$\bar{x} \sim N(m, \frac{s}{\sqrt{n}})$$

make an $\bar{x}$ value

into a z-score.

The general z-score

expression is

$$z = \frac{(x - m)}{s}$$

for $\bar{x}$,

$m_{\bar{x}}$ is m : *unchanged*

and

$s_{\bar{x}}$ is $\frac{s}{\sqrt{n}}$

so a z-score

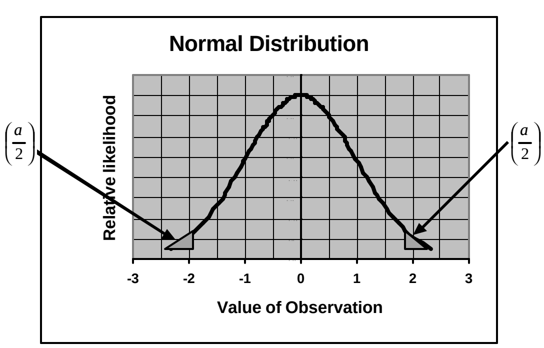
based on $\bar{x}$ is

$$z = \frac{\bar{x} - m}{s/\sqrt{n}}$$

Using the Empirical Rule

Make a probability statement:

$$P\left(-2 < \frac{(\bar{x} - m)}{s/\sqrt{n}} < 2\right) = 95\%$$



Check out the "Confidence z-scores" on the WEB page.

(In pdf format.)

Use basic rules of algebra to rearrange the parts of this z-score.

Manipulate the probability statement:

$$P\left(-2\left(s\big/\sqrt{n}\right) < \left(\overline{x} - m\right) < 2\left(s\big/\sqrt{n}\right)\right) = 0.95$$

Manipulate the probability statement:

$$P\left(-\bar{x}-2\left(s/\sqrt{n}\right)<-m<-\bar{x}+2\left(s/\sqrt{n}\right)\right)=0.95$$

Confidence = 95%

a **= 1 - 95% = 5%**

a/2 = **2.5% = 0.025**

Manipulate the probability statement:
multiply through by (-1) and change the order of the terms

$$P\left(\bar{x}-2\left(s/\sqrt{n}\right)<m<\bar{x}+2\left(s/\sqrt{n}\right)\right)=0.95$$

Confidence = 95%

a **= 1 - 95% = 5%**

a/2 = **2.5% = 0.025**

Confidence Interval for m

If s **is not known** **(usual situation)**

$$=\bar{x}\pm t_{a/2}\cdot\left(\frac{s}{\sqrt{n}}\right)$$

Sample Size Needed to Estimate m within E, with Confidence = 1-a

$$n = \left[\frac{Z_{a/2} \cdot \hat{s}}{E} \right]^2$$

Components of Sample Size Formula when Estimating m

- **$Z_{a/2}$ reflects confidence level**
  - **standard normal distribution**
- **$\hat{s}$ is an estimate of s, the standard deviation of the pop.**
- **E is the acceptable “margin of error” when estimating m**

Confidence Interval for p

- **The Binomial Distribution gives us a starting point for determining the distribution of the sample proportion : $\hat{p}$**

$$\hat{p} = \frac{x}{n} = \frac{successes}{trials}$$

For Binomial "x"

$$m = np$$

$$s = \sqrt{npq}$$

For the Sample Proportion

$$\hat{p} = \frac{x}{n} = \frac{1}{n}(x)$$

x is a random <u>variable</u>
n is a <u>constant</u>

<u>Time Out</u> for a Principle:

If m is the mean of X and "a" is a <u>constant</u>, what is the mean of aX?

Answer: $a \cdot m$

Apply that Principle!

- **Let "a" be equal to "1/n"**
- **so** $\hat{p} = aX = \left(\frac{1}{n}\right)X = \frac{X}{n}$
- **and** $m_{\hat{p}} = am_x = a(np)$

  $= \left(\frac{1}{n}\right) \cdot np = p$

Time Out for another Principle:

If s_x^2 is the variance of X and "a" is a constant, what is the variance of aX?

Answer: $s_{aX}^2 = a^2 s_x^2$

Apply that Principle!

- **Let x be the binomial "x"**
- **Its variance is npq = np(1-p), which is the square of is standard deviation**

Apply that Principle!

- **Let "a" be equal to "1/n"**
- **so** $\hat{p} = aX = \left(\frac{1}{n}\right)X = \frac{X}{n}$
- **and** $s_{\hat{p}}^{2} = a^2 s_X^{2} = (1/n)^2 (npq)$

Apply that Principle!

$$\left(\frac{1}{n}\right)^2 \cdot npq = \frac{pq}{n} = s_{\hat{p}}^{2}$$

$$and$$

$$s_{\hat{p}} = \sqrt{\frac{pq}{n}}$$

When n is Large,

$$\hat{p} \sim N\left(m = p, s = \sqrt{\frac{pq}{n}}\right)$$

What is a Large "n" in this situation?

- **Large enough so np > 5**
- **Large enough so n(1-p) > 5**
- **Examples:**
  - **(100)(0.04) = 4 (too small)**
  - **(1000)(0.01) = 10 (big enough)**

Now make a z-score

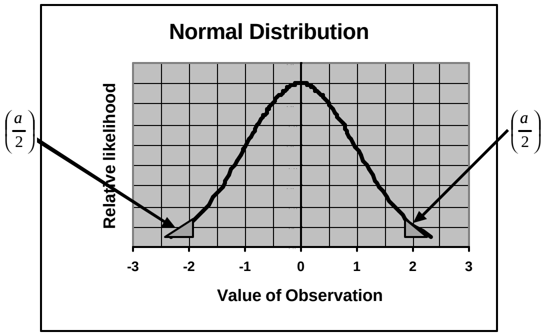
$$z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}}$$

And rearrange for a CI(p)

Using the Empirical Rule

Make a probability statement:

$$P\left(-1.96 < \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}} < 1.96 \right) = 95\%$$



Use basic rules of algebra to rearrange the parts of this z-score.

Manipulate the probability statement:

Step 1: Multiply through by $\sqrt{\frac{pq}{n}}$:

$$P\left(-1.96\sqrt{\frac{pq}{n}} < (\hat{p} - p) < 1.96\sqrt{\frac{pq}{n}}\right) = 0.95$$

Manipulate the probability statement:

Step 2: Subract $\hat{p}$ from all parts of the expression:

$$P\left(-\hat{p}-1.96\sqrt{\frac{pq}{n}}<-p<-\hat{p}+1.96\sqrt{\frac{pq}{n}}\right)=0.95$$

Manipulate the probability statement:

Step 3: Multiply through by -1:

(remember to switch the directions of < >)

$$P\left(\hat{p}+1.96\sqrt{\frac{pq}{n}}>p>\hat{p}-1.96\sqrt{\frac{pq}{n}}\right)=0.95$$

Manipulate the probability statement:

Step 4: Swap the left and right sides to put in conventional < p < form:

$$P\left(\hat{p}-1.96\sqrt{\frac{pq}{n}}<p<\hat{p}+1.96\sqrt{\frac{pq}{n}}\right)=0.95$$

Confidence Interval for p
(but the unknown p is in the formula. What can we do?)

$$= \hat{p} \pm z_{a/2} \cdot \sqrt{\frac{pq}{n}}$$

Confidence Interval for p
(substitute sample statistic for p)

$$= \hat{p} \pm z_{a/2} \cdot \sqrt{\frac{\hat{p}\hat{q}}{n}}$$

Sample Size Needed to Estimate "p" within E, with Confid.=1- a

$$n = \left(\frac{Z_{a/2}^2}{E^2} \right) \cdot \hat{p}\hat{q}$$

Components of Sample Size Formula when Estimating "p"

- $Z_{a/2}$ is based on a using the standard normal distribution
- p and q are estimates of the population proportions of "successes" and "failures"
- E is the acceptable "margin of error" when estimating m

Components of Sample Size Formula when Estimating "p"

- p and q are estimates of the population proportions of "successes" and "failures"
- Use relevant information to estimate p and q if available
- Otherwise, use p = q = 0.5, so the product pq = 0.25

Confidence Interval for s starts with this fact

$$\text{if } x \sim N(m, s)$$

then

$$\frac{(n-1)s^2}{s^2} \sim c^2 \text{ (chi square)}$$

What have we studied already that connects with Chi-square random values?

$$\frac{(n-1)s^2}{s^2} \sim c^2 \text{ (chi square)}$$

$$\frac{(n-1)s^2}{s^2} = \frac{(n-1)\frac{\sum (x-m)^2}{(n-1)}}{s^2}$$

$$= \frac{\sum (x-m)^2}{s^2}$$

$$\sum\left[\frac{(x-m)^2}{s^2}\right] = \sum\left[\frac{(x-m)}{s}\right]^2$$

$$= \sum z^2$$ a sum of squared standard normal values

Confidence Interval for s

$$LB = \sqrt{\frac{(n-1)s^2}{c_R^2}}$$

$$UB = \sqrt{\frac{(n-1)s^2}{c_L^2}}$$